

# 一些证明

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1. 求和:  $S = \sum_{i=1}^n i^2$

证明.

$$\begin{aligned}\because n^3 &= \sum_{i=1}^n [i^3 - (i-1)^3] \\ &= \sum_{i=1}^n (3 * i^2 - 3 * i + 1) \\ &= 3 \sum_{i=1}^n i^2 - 3 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 3S - 3 \frac{n(n+1)}{2} + n \\ \therefore 3S &= n^3 + \frac{3n(n+1)}{2} - n \\ \therefore S &= \frac{2n^3 + 3n^2 + 1}{6} \\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

□

2. 求和:  $S = \sum_{i=1}^n i^3$

证明.

$$\begin{aligned}
 \because n^4 &= \sum_{i=1}^n [i^4 - (i-1)^4] \\
 &= \sum_{i=1}^n (4i^3 - 6i^2 + 4i - 1) \\
 &= 4 \sum_{i=1}^n i^3 - 6 \sum_{i=1}^n i^2 + 4 \sum_{i=1}^n i - n \\
 &= 4S - 6 \times \frac{n(n+1)(2n+1)}{6} + 4 \times \frac{n(n+1)}{2} - n \\
 \therefore 4S &= n^4 - n(n+1)(2n+1) + 2n(n+1) - n \\
 \therefore S &= \frac{n^2(n+1)^2}{4} \quad \square
 \end{aligned}$$

3. 已知:  $abc = 1, a, b, c \in R$

求证:  $(a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) \leq 1$

证明.

$$\begin{aligned}
 (a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) &= bc(a - 1 + \frac{1}{b}) \times ac(b - 1 + \frac{1}{c}) \times ab(c - 1 + \frac{1}{a}) \\
 &= (1 - bc + c)(1 - ca + a)(1 - ab + b) \\
 &= (1 - \frac{1}{a} + c)(1 - \frac{1}{b} + a)(1 - \frac{1}{c} + b)
 \end{aligned}$$

$$\left. \begin{aligned}
 (a - 1 + \frac{1}{b})(1 - \frac{1}{b} + a) &\leq \left[ \frac{(a - 1 + \frac{1}{b}) + (1 - \frac{1}{b} + a)}{2} \right]^2 = a^2 \\
 (b - 1 + \frac{1}{c})(1 - \frac{1}{c} + b) &\leq \left[ \frac{(b - 1 + \frac{1}{c}) + (1 - \frac{1}{c} + b)}{2} \right]^2 = b^2 \\
 (c - 1 + \frac{1}{a})(1 - \frac{1}{a} + c) &\leq \left[ \frac{(c - 1 + \frac{1}{a}) + (1 - \frac{1}{a} + c)}{2} \right]^2 = c^2
 \end{aligned} \right\}$$

$$\begin{aligned}
 \Rightarrow (a - 1 + \frac{1}{b})(1 - \frac{1}{b} + a)(b - 1 + \frac{1}{c})(1 - \frac{1}{c} + b)(c - 1 + \frac{1}{a})(1 - \frac{1}{a} + c) &\leq a^2 b^2 c^2 = 1 \\
 \Rightarrow (a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) &\leq \sqrt{1} = 1 \quad \square
 \end{aligned}$$

4. 如果  $\lim_{n \rightarrow \infty} a_n = a$ , 则  $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = a$

证法一

证明. 根据数列极限的定义来证明: 假设  $a=0$ , 即  $\lim_{n \rightarrow \infty} a_n = 0$ , 根据数列极限的定义, 有  $\forall \epsilon > 0, \exists N_1$ , 当  $n \leq N_1$  时, 有  $|a_n| < \frac{\epsilon}{2}$ , 于是

$$\begin{aligned} \left| \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} \right| &\leq \frac{|a_1| + |a_2| + |a_3| + \dots + |a_n|}{n} \\ &= \frac{|a_1| + |a_2| + |a_3| + \dots + |a_{N_1}|}{n} + \frac{(n - N_1)\epsilon}{2n} \end{aligned}$$

下面要证明上式的右端小于  $\epsilon$ 。由于  $|a_1| + |a_2| + |a_3| + \dots + |a_{N_1}|$  是定数, 所以  $\exists N_2$ , 使得当  $n > N_2$  时有  $\frac{|a_1| + |a_2| + |a_3| + \dots + |a_{N_1}|}{n} < \frac{\epsilon}{2}$ 。所以当  $n > \max N_1, N_2$  时,

$$\frac{|a_1| + |a_2| + |a_3| + \dots + |a_{N_1}|}{n} + \frac{(n - N_1)\epsilon}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, 于是根据极限的定义即  $\lim_{n \rightarrow \infty} a_n = 0$ 。当  $a \neq 0$  时, 令  $b_n = a_n - a$ , 则可由上面的结论得

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} - a \right) &= \lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n - na}{n} \\ &= \lim_{n \rightarrow \infty} \frac{b_1 + b_2 + b_3 + \dots + b_n}{n} = \lim_{n \rightarrow \infty} b_n = 0 \end{aligned}$$

于是

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = a$$

□

证法二

证明. 令  $x_n = a_1 + a_2 + a_3 + \dots + a_n$ ,  $y_n = n$ , 显然数列  $y_n$  是单调递增的无穷大量, 所以代人 Stolz 公式可得

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n \rightarrow \infty} \frac{a_n}{1} = a$$

□

5. 如果  $\lim_{n \rightarrow \infty} a_n = a$  且  $a_n \geq 0$ , 则  $\lim_{n \rightarrow \infty} \sqrt[n]{a_1 a_2 a_3 \dots a_n} = a$

证明. 因为  $\lim_{n \rightarrow \infty} a_n = a$ , 所以  $\lim_{n \rightarrow \infty} \ln a_n = \ln a$ , 又因为  $e^x$  在  $\mathbb{R}$  上连续, 所以

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{a_1 a_2 a_3 \dots a_n} &= \lim_{n \rightarrow \infty} e^{\frac{1}{n}(\ln a_1 + \ln a_2 + \ln a_3 + \dots + \ln a_n)} \\ &= e^{\lim_{n \rightarrow \infty} \frac{1}{n}(\ln a_1 + \ln a_2 + \ln a_3 + \dots + \ln a_n)} = e^{\ln a} = a \end{aligned}$$

□

6. 求极限:  $\lim_{n \rightarrow \infty} \sqrt[n]{n}$

证明. 令  $a_n = \frac{n}{n-1}$  (令  $a_1 = 1$ ), 应用上题的结论可得  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2}{1} \frac{3}{2} \frac{4}{3} \dots \frac{n}{n-1}} = \lim_{n \rightarrow \infty} \frac{n}{n-1} = 1$

□

7. 如果  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} b_n = b$ , 则  $\lim_{n \rightarrow \infty} \frac{a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \dots + a_n b_1}{n} = ab$   
 说明: 这个问题是  $\lim_{n \rightarrow \infty} a_n = a \Rightarrow \lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = a$  的推广。

证明. 证明: 假设  $a=0$  并且  $b=0$ 。由  $a_n$  收敛到 0 可知  $a_n$  有界, 即  $\exists M > 0$ , 使得  $|a_n| \leq M (\forall n \in \mathbb{N})$ 。

$$0 < \left| \frac{a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \dots + a_n b_1}{n} \right| \leq M \frac{b_n + b_{n-1} + b_{n-2} + \dots + b_1}{n} \rightarrow 0$$

当  $ab \neq 0$  时, 令  $x_n = a_n - a, y_n = b_n - b$ , 配合  $\lim_{n \rightarrow \infty} a_n = a \Rightarrow \lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = a$  可轻松得到结论。

□